ENTROPY LAYER IN TWO-DIMENSIONAL FLOWS

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We consider plane and axisymmetric hypersonic flows of an ideal gas behind a compression shock satisfying the conditions $\cos(\mathbf{n}, \mathbf{i}) \leqslant \tau$, $\cos(\mathbf{n}, \mathbf{j}) \sim 1$ everywhere except for a small vicinity of the apex. Here the small parameter $\tau \ll 1$; $\mathbf{n}$ is the normal to the surface of the front; $\mathbf{i}$ is the unit vector of the x_1 axis; $\mathbf{j}$ is the unit vector of the x_2 axis; Lx_i $(i = 1, 2)$ is the rectangular Cartesian coordinate system; L is the characteristic length. The impinging stream is assumed to be homogeneous. Assuming that the condition $K = M_\infty \tau \geqslant 1$ is satisfied for a shock having close to a power-law form close to the apex, we consider the behavior of the solution outside the vicinity of the apex as $\tau \to 0$ on the basis of the method of joined asymptotic expansions. In doing this we take the expansion of the hypersonic theory of small perturbations [1], which describes unsteady one-dimensional flows in the zeroth approximation, as the outer expansion in the region adjacent to the front.

A great many results in the theory of the flow over thin bodies by a hypersonic stream are based on an analogy with unsteady flows in a space with a number of dimensions smaller by one. However, the validity of such an analogy in the case when an entropy layer develops in the stream has not yet been fully proven [2]. A detailed analysis of the known results on this question is carried out in [3, 4]. An explicit connection between the two-dimensional problem under consideration and the unsteady one-dimensional problem is established in the present report. Particular cases of the results obtained are compared with the known results.

The system of gasdynamic equations in the variables "pressure — two stream functions" [5] has the following form in the two-dimensional case:

$$\frac{\partial u_i}{\partial p} = (-1)^i x_2^{\nu} \frac{\partial x_{i+1}}{\partial \psi}; \quad \frac{\partial x_1}{\partial p} u_2 = \frac{\partial x_2}{\partial p} u_1;$$
$$\frac{u_1^2 + u_2^2}{2} + \frac{\varkappa}{\varkappa - 1} \frac{p}{\rho} = \frac{1}{2} + \frac{1}{\varkappa - 1} M_\infty^{-2}; \quad \frac{\partial}{\partial p}\left(\frac{p}{\rho^\varkappa}\right) = 0. \tag{1}$$

Here and below an index $i > 2$ is read as $i - 2$; $\nu = 0$ and 1 for the plane and axisymmetric cases; $\rho_\infty u^2_\infty p$ is the pressure; $\rho_\infty \rho$ is the density; $u_\infty u_i$ are the components of the velocity vector along the x_i axes, respectively; $\rho_\infty u_\infty L^{1+\nu}\psi$ is the stream function; the x_1 axis is directed along the velocity vector u_∞ of the undisturbed stream; and κ is the ratio of specific heat capacities of the gas. Henceforth we will assume that the condition $1 < \kappa < 2$ is satisfied. The index ∞ pertains to the conditions in the undisturbed stream.

The boundary conditions at the shock wave have the form

$$p = \frac{2}{\varkappa + 1}(\mathbf{i}\cdot\mathbf{n})^2 + \frac{1-\varkappa}{\varkappa(\varkappa+1)} M_\infty^{-2}; \quad \frac{1}{\rho} = \frac{\varkappa - 1}{\varkappa + 1} + \frac{2}{\varkappa + 1} M_\infty^{-2} (\mathbf{i}\cdot\mathbf{n})^{-2};$$
$$(\mathbf{i} - \mathbf{v})(\mathbf{i}\cdot\mathbf{n}) = \left(p - \frac{1}{\varkappa} M_\infty^{-2}\right)\mathbf{n},$$

where $\mathbf{v}$ is the velocity vector of the disturbed motion at the shock. Being confined for simplicity to a surface of a front which is symmetrical relative to the x_1 axis, we assign the analytical expression for the shape of the front in the form

$$x_2 = \tau 2^{\nu/2} \zeta^{1/(1+\nu)}, \quad x_1 = g(\zeta) \quad \text{for} \quad x_2 \geqslant 0,$$

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where the function $g(\zeta)$ provides for a smooth solution in the disturbed region and has the following properties: 1) $\lim g(\zeta)/\zeta^{\alpha} = \text{const}$ as $\zeta \to 0$; $1/(1+\nu) < \alpha \leqslant (3+\nu)/2(1+\nu)$; 2) $dx_2/dx_1 \leqslant 0(\tau)$ everywhere except for a small vicinity of the apex. Here the limitation $\alpha > 1/(1+\nu)$ provides for the appearance of an entropy layer, while $\alpha \leqslant (3+\nu)/2(1+\nu)$ is a limitation emerging from the mean of construction of the solution outside the entropy layer. The first property assumes that the form of the front is close to a power law at the apex. Outside the vicinity of the apex in the region where $p \leqslant 0(\tau^2)$ the solution is constructed by the method of joined asymptotic expansions [6].

In the inner region adjacent to the streamline $\zeta = 0$ the inner expansion is constructed as follows. We consider the entropy layer, i.e., the gas layer passing through the section of the shock wave with a slope $1 > \beta \geqslant 0$, $\tau^{\beta} \leqslant \cos(\mathbf{n}, \mathbf{i}) \leqslant 1$. With allowance for the law of conservation of entropy per particle, as well as the order of magnitude of the pressure, we obtain in the entropy layer $0(\tau^{2/\varkappa}) \leqslant \rho \leqslant 0(\tau^{(2-2\beta)/\varkappa})$. Henceforth we set $\rho \sim 0(\tau^{\gamma})$ in the entropy layer, where $2/\kappa \geqslant \gamma \geqslant (2-2\beta)/\kappa$. At the boundary of the region under consideration $u_2 \leqslant 0(\tau)$. Assuming that the condition $u_2 \leqslant 0(\tau)$ is satisfied across the region under consideration, we obtain from the Bernoulli integral

$$u_1 \sim 1 + 0(\tau^{2-\gamma}). \tag{2}$$

Thus,

$$u_1 = f(1 + 0(\tau^2)),\ f = \{1 - [2\varkappa/(\varkappa - 1)]p/\rho\}^{1/2}. \tag{3}$$

And for the derivative of the Bernoulli integral with allowance for (2) we get

$$\partial u_1/\partial p = -(1/\rho f)(1 + 0(\tau^{\gamma})). \tag{4}$$

Across the disturbed region γ varies from $2/\kappa$ in the vicinity of the surface of the body to 0 in the vicinity of the surface of the front. A concept of the region of applicability of (4) can be obtained by considering the isobars $p = \text{const}$. For example, everywhere in the region under consideration where $|\partial p/\partial x_1/\partial p/\partial x_2| \geqslant 0(1)$, the parameter $\gamma \geqslant 1$.

The fulfillment of Eq. (4) in the entropy layer allows one to considerably simplify system (1). Substituting (3) and (4) into (1), we obtain

$$\begin{gathered}\left(\frac{x_2}{\tau}\right)^{1+\nu} = 2^{\nu}\int_0^{\zeta}\frac{1}{\rho f}\,d\zeta\,(1 + 0(\tau^{\gamma})) + x_{20}^{1+\nu}(\xi, \tau);\\ \frac{\partial}{\partial\xi}\left[\frac{\partial}{\partial\xi}\left(\frac{x_2}{\tau}\right)\left(\frac{\partial x_1}{\partial\xi}\right)^{-1} f\right](1 + 0(\tau^{\gamma})) = \left(\frac{x_2}{\tau}\right)^{\nu}\frac{\partial x_1}{\partial\zeta};\\ \frac{u_2}{\tau}\frac{\partial x_1}{\partial\xi} = f\frac{\partial}{\partial\xi}\left(\frac{x_2}{\tau}\right)(1 + 0(\tau^2)),\ \ \xi/\rho^{\varkappa} = F(\tau, \zeta),\end{gathered} \tag{5}$$

where $p = \tau^2\xi$; $\psi = \tau^{1+\gamma}\zeta$; x_{20} and F are arbitrary functions. In the coordinate system used the pressure enters into the integrand as a parameter. Therefore, the integration of system (1), if the entropy distribution along the streamlines is known, comes down to the integration of the one quasilinear parabolic equation with respect to x_1 in system (5) in the entropy layer.

Representing x_1 for small ζ in the form of asymptotic expansions with respect to ζ, we obtain from the second equation of system (5)

$$x_1 = x_{10}(\xi, \tau) + x_{11}(\xi, \tau)\,\zeta + \varphi(\xi, \tau, \zeta), \tag{6}$$

where the function x_{10} is arbitrary;

$$x_{11} = \begin{cases} x_{20}^{-\nu}\dfrac{d}{d\xi}\left[\dfrac{d}{d\xi}(x_{20})\dfrac{dx_{10}^{-1}}{d\xi}\right], & \text{if} \quad x_{20} \neq 0,\\ \nu\xi^{\frac{1}{2\varkappa}}\dfrac{d}{d\xi}\left[\dfrac{d}{d\xi}\left(\xi^{-\frac{1}{2\varkappa}}\right)\dfrac{dx_{10}^{-1}}{d\xi}\right], & \text{if} \quad x_{20} = 0\,. \end{cases}$$

The function φ does not exceed the following quantities in order of magnitude:

$$\varphi = \begin{cases} O\left(\tau^{2-\frac{2}{\varkappa}}\zeta + \tau^{\gamma}\zeta + \int_0^{\zeta}\int \frac{1}{\rho}d\zeta d\zeta\right), & \text{if} \quad x_{20} \neq 0, \\ O\left(\int_0^{\zeta}\int \frac{1}{\rho}d\zeta d\zeta\right) \ \text{at} \ \nu = 0; \ O\left(\tau^{2-\frac{2}{\varkappa}}\zeta + \tau^{\gamma}\zeta\right) \\ \text{at} \ \nu = 1, \ \text{if} \quad x_{20} = 0. \end{cases}$$

The approximate solution (3)-(6) constructed depends on the three arbitrary functions $x_{20}(\xi,\tau)$, $x_{10}(\xi, \tau)$, and $F(\tau, \zeta)$, which describe the surface of the body, the pressure distribution over the body, and the entropy distribution along the streamlines, respectively. We represent the solution in the region adjacent to the front in the form

$$\begin{aligned} x_i &= \tau^{i-1}(y_i + \tau^2 z_i + \ldots \quad u_1 = 1 + \tau^2 v_1 + \tau^4 w_1 + \ldots \\ \rho &= \rho_0 + \tau^2\rho_1 + \ldots \quad u_2 = \tau(v_2 + \tau^2 w_2 + \ldots \\ p &= \tau^2(\xi + \tau^2 p_1 + \ldots \quad \psi = \tau^{1+\nu}\zeta. \end{aligned} \tag{7}$$

The expansions (7) are obtained in accordance with the orders of magnitude of the unknown quantities at the front. The functions p_i are given only through their values at the front, for the satisfactorion of which it is sufficient to take $p_i = p_i(\xi)$. Changing to the new independent variables ξ and ζ, after the substitution of (7) into (1) we obtain

$$\begin{gathered} \frac{\partial v_i}{\partial \xi} = (-1)^i y_2^{\nu}\frac{\partial y_{i+1}}{\partial \zeta}; \quad \frac{\partial}{\partial \xi}\left[\frac{\xi}{\rho_0^{\varkappa}}\right] = 0; \\ \frac{\partial w_i}{\partial \xi} - \frac{\partial p_1}{\partial \xi}\frac{\partial v_i}{\partial \xi} = (-1)^i\left[y_2^{\nu}\frac{\partial z_{i+1}}{\partial \zeta} + \nu\frac{\partial y_{i+1}}{\partial \zeta}z_2\right]; \quad \frac{\partial}{\partial \xi}\left[\frac{p_1}{\xi} - \varkappa\frac{\rho_1}{\rho_0}\right] = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_1 + \frac{1}{2}v_2^2 + \frac{\varkappa}{\varkappa - 1}\frac{\xi}{\rho_0} = \frac{1}{\varkappa - 1}K^{-2}; \\ w_1 + \frac{1}{2}v_1^2 + v_2 w_2 + \frac{1}{2}w_2^2 + \frac{\varkappa}{\varkappa - 1}\left(\frac{p_1}{\rho_0} - \frac{\xi\rho_1}{\rho_0^2}\right) = 0; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{\partial y_1}{\partial \xi}v_2 = \frac{\partial y_2}{\partial \xi}; \quad \frac{\partial z_1}{\partial \xi}v_2 + \frac{\partial y_1}{\partial \xi}w_2 = \frac{\partial z_2}{\partial \xi} + v_1\frac{\partial y_2}{\partial \xi}. \end{gathered} \tag{8}$$

The expression for the normal to the front has the form

$$\begin{gathered} n_1 = -\tau A(1 + \tau^2 A^2)^{-1/2}; \ n_2 = (1 + \tau^2 A^2)^{-1/2}; \\ A = 2^{-\nu/2}\ \zeta^{-\nu/(1+\nu)}dg^{-1}/d\zeta, \end{gathered}$$

and by expanding this in series which converge for $\tau^2 A^2 < 1$ and substituting the expansions into the boundary conditions at the front, we obtain

$$\begin{gathered} \xi = [2/(\varkappa + 1)]A^2 + [(1 - \varkappa)/\varkappa(\varkappa + 1)]K^{-2}; \\ v_1 = -[2/(\varkappa + 1)]A^2 + [2/(\varkappa + 1)]K^{-2}; \\ p_1 = -[2/(\varkappa + 1)]A^4; \ w_1 = [2/(\varkappa + 1)]A^4; \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 1/\rho_0 = (\varkappa - 1)/(\varkappa + 1) + [2/(\varkappa + 1)]K^{-2}A^{-2}; \\ v_2 = \{[2/(\varkappa + 1)]A^2 - [2/(\varkappa + 1)]K^{-2}\}A^{-1}; \\ \frac{\rho_1}{\rho_0^2} = -\frac{2}{\varkappa + 1}K^{-2}; \quad w_2 = -\frac{2}{\varkappa + 1}A^3. \end{gathered} \tag{9}$$

Considering the condition $\tau \to 0$, we will assume that the boundary conditions (9) are satisfied for $\zeta \geqslant 0$. Thus, the problem becomes closed for each approximation of the outer expansion (7). The solution of the zeroth approximation describes plane and axisymmetric one-dimensional unsteady gas flows behind a front of the form $t = g(\zeta)$, $r = 2^{\nu/2}\zeta^{1/(1+\nu)}$, where t is the time; r is the spatial coordinate. It is shown in [7] that for the existence of a solution of the unsteady problem it is necessary that $\alpha \leqslant (3+\nu)/2(1+\nu)$. From this follows the limitation imposed above on α.

The joining of (3)–(6) only with the zeroth approximation of the expansion (7) is carried out below. Since the zeroth approximation itself is an asymptotic representation of the exact solution outside the inner region, the terms of the zeroth approximation retained in the joining must be of a lower order of smallness than the principal terms of the higher approximations. And the choice of the region of joining is not obvious without an analysis of the behavior of the higher approximations. Therefore, it seems appropriate to make a brief study of the behavior of the higher approximations.

If in a region R adjacent to the front, except for some vicinity of the lines $\zeta = 0$, the boundary conditions provide for a sufficiently smooth solution of the zeroth approximation, then the continuous solution of the higher approximations up to some number j is determined in this region. To discover the nature of the behavior of the latter in the vicinity of the line $\zeta = 0$ we represent the solution of the k-th approximation in the form of the asymptotic expansions

$$F_k = F_{k0}(\xi)g_{k0}(\zeta) + F_{k1}(\xi)g_{k1}(\zeta) + \ldots, \quad g_{k,j+1}/g_{k,j} \to 0 \text{ as } \zeta \to 0.$$

Substituting the latter into (8) and allowing for the conditions (9), we obtain

$$\rho \sim \Delta^{-2}\Phi;\ u_1 \sim 1 + \tau^2\Delta^2\Phi;\ x_1 \sim 0(1) + \tau^2\Delta^2\zeta\Phi;$$
$$x_2 \sim \tau(0(1) + \tau^2\zeta\Delta^{2+2\varkappa}\Phi);\ u_2 \sim \tau(0(1) + \tau^2\Delta^2\Phi),$$

if $y_{20} \neq 0$. In the case of $y_{20} = 0$ the only difference will be with $\nu = 1$ for the components

$$x_1 \sim 0(1) + \tau^2 0\left(\Delta\zeta^{1/2},\ \int_0^\zeta \Delta^{2\varkappa} d\zeta\right)\Phi; \quad x_2 \sim \tau\Delta\zeta^{1/2}\Phi;$$
$$u_2 \sim \tau\Delta\zeta^{1/2}(0(1) + \tau^2\Delta^{2\varkappa}\Phi);\ \Phi = 0(1) + \tau^2 0(\Delta^{2\varkappa}) + \tau^4 0(\Delta^{4\varkappa}) + \ldots$$

The term $\tau^{2j}0(\Delta^{2\kappa j})$, where $\Delta = \zeta^{1/\kappa[1/(1+\nu)-\alpha]}$, in the expression for the function Φ describes the nature of the behavior of the j-th approximation in the vicinity of the line $\zeta = 0$ relative to the zeroth approximation. Thus, the rate of decline of the asymptotic expansions with respect to τ is characterized by the expression $\tau^2 0(\Delta^{2\kappa}) = (\zeta^{2[1/(1+\nu)-\alpha]})$. Therefore, the corrections to the zeroth approximation are generally not small in the region of $\tau^2 0(\Delta^{2\kappa}) \sim 1$, for which the values of the entropy in the zeroth approximation are close to the values of the entropy behind a straight compression shock in a steady stream. The fact of the necessity of a correction to the zeroth approximation for these values of ζ is obtained from other considerations in [3].

Being confined to only the zeroth approximation of the outer expansion, we determined the arbitrary functions x_{20} and x_{10} of (5) and (6) from the condition of joining in the region $\tau^2 0(\Delta^{2\kappa}) \sim \tau^{\beta_0}$, where the zeroth approximation and (5) have the same accuracy. Using the boundary conditions at the front, one can show that in this case $\beta_0 = 2/(\kappa + 1)$ and one must set $\gamma = \beta_0$ in (5) and (6), while the region of joining is determined by the values of $\zeta \sim 0(\tau^{\varkappa(1+\nu)/(\varkappa+1)(\alpha(1+\nu)-1)})$. The solution of the zeroth approximation for the ζ of this region has the form

$$y_2^{1+\nu} = y_{20}^{1+\nu}(\xi) + 2^\nu \int_0^\zeta \frac{1}{\rho_0} d\zeta\,(1 + 0(\tau^{\beta_0}));$$

$$v_2 = \frac{dy_{10}^{-1}}{d\xi}\frac{\partial}{\partial\xi}\left[y_{20}^{1+\nu}(\xi) + 2^\nu \int_0^\zeta \frac{1}{\rho_0} d\zeta\,(1 + 0(\tau^{\beta_0}))\right]^{\frac{1}{1+\nu}}(1 + 0(\tau^{\beta_0}));$$

$$
y_1 = \begin{cases} y_{10}(\xi) + y_{20}^{-\nu} \dfrac{d}{d\xi}\left[\dfrac{dy_{10}^{-1}}{d\xi}\dfrac{dy_{20}}{d\xi}\right]\zeta\left(1 + 0\left(\tau^{\beta_0}\right)\right) + 0\left(\displaystyle\int_0^{\xi}\int \frac{1}{\rho_0}\,d\zeta d\zeta\right), & y_{20} \neq 0, \\ y_{10}(\xi) + \nu\xi^{\frac{1}{2\varkappa}} \dfrac{d}{d\xi}\left[\dfrac{dy_{10}^{-1}}{d\xi}\dfrac{d}{d\xi}\xi^{-\frac{1}{2\varkappa}}\right]\zeta\left(1 + 0\left(\tau^{\beta_0}\right)\right) + (1-\nu)\,0\left(\displaystyle\int_0^{\xi}\int \frac{1}{\rho_0}\,d\zeta d\zeta\right), & y_{20} = 0; \end{cases}
$$

$$
v_1 = -\frac{\varkappa}{\varkappa - 1}\frac{\xi}{\rho_0} + 0(1), \quad \frac{1}{\rho_0} = c_0\xi^{-\frac{1}{\varkappa}} A^{\frac{2}{\varkappa}}, \quad c_0 = \left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa}}\left(\frac{\varkappa-1}{\varkappa+1}\right).
$$

Here we retain only the terms which are of a lower order of smallness than the principal terms of the higher approximations. The function $t = y_{10}(\xi)$ describes the pressure distribution, while the function $r = y_{20}(\xi)$ describes the trajectory of a particle passing through the front at the start of the motion. The methods of determining the latter are well developed [1, 8], and therefore we will henceforth consider the functions y_{10} and y_{20} as known. For a detailed analysis of the flow we require the concrete form of the function $g(\zeta)$ in the small vicinity of the apex of the front. We perform the analysis for functions $g(\zeta)$ which for $\zeta \leqslant 0(\tau^{\varkappa(1+\nu)/(\varkappa+1)(\alpha(1+\nu)-1)})$ satisfy the condition

$$
A = A_0\left(1 + 0\left(\tau^{\beta_0}\right)\right), \quad A_0 = 2^{-\nu/2}(c\alpha)^{-1}\zeta^{1/(1+\nu)-\alpha}, \tag{10}
$$

which clarifies the degree of closeness of the function $g(\zeta)$ to a power-law function. Here $c > 0$ is an arbitrary constant. Since $K \geqslant 1$, from the boundary conditions at the front for the indicated values of ζ we have

$$
\frac{1}{\rho} = c_0\xi^{-\frac{1}{\varkappa}}\tau^{-\frac{2}{\varkappa}} X\left(1 + 0\left(\tau^{\beta_0}\right)\right), \quad X = \left(\frac{\tau^2 A_0^2}{1 + \tau^2 A_0^2}\right)^{\frac{1}{\varkappa}}.
$$

Using (10), one can obtain

$$
\int_0^{\zeta} \frac{1}{\rho f}\,d\zeta = c_0\xi^{-\frac{1}{\varkappa}}\tau^{-\frac{2}{\varkappa}} \int_0^{\zeta} X\left(1 + \sum_{j=1}^{l} \frac{\eta(\eta-1)\ldots(\eta-j+1)}{j!}\left(B\tau^{2-\frac{2}{\varkappa}}X\right)^j\right) d\zeta\left(1 + 0\left(\tau^{\beta_0}\right)\right), \tag{11}
$$

where $B = -[2\kappa/(\kappa-1)]c_0\xi^{1-1/\kappa}$; $\eta = -1/2$; the integer l is determined below. In the case of $\alpha = (3+\nu)/2(1+\nu)$ (a strong explosion in the corresponding unsteady problem with subsequent expansion of the piston) we have

$$
\int_0^{\zeta} X^n d\zeta = \frac{\varkappa}{\varkappa - n}\mu\left(X^{n-\varkappa} - 1\right), \quad \mu = c^{-2}\left(\frac{4}{9}\right)^{1-\nu} 2^{-\nu}\tau^2, \quad n = 1, 2, \ldots. \tag{12}
$$

In the general case for ζ from the region of joining we have

$$
\int_0^{\zeta} X^n d\zeta = \frac{1+\nu}{2(\alpha(1+\nu)-1)}\left(\tau F_0\right)^{\frac{1+\nu}{\alpha(1+\nu)-1}}\left(\Phi_1(b) - \Phi_0(\zeta)\right). \tag{13}
$$

Here and below $F_0 = 2^{-\nu/2}(c\alpha)^{-1}$, $b = -n/\kappa$, $a = -b - h - 2$, $h = (1+\nu)/2(\alpha(1+\nu)-1) - 1$; the function Φ_0 has the form

$$
\begin{cases} 0\left(\tau^2 A_0^2\right), & a + 1 \neq 0, \\ 0\ln\left(\tau^2 A_0^2\right), & a + 1 = 0, \end{cases} \quad \text{if} \quad 1/\varkappa < -b,
$$

$$
\text{and } (a+1)^{-1}\left(\tau^2 A_0^2\right)^{a+1} + 0\left((a+2)^{-1}\left(\tau^2 A_0^2\right)^{a+2}\right) \text{ when } b = -1/\varkappa.
$$

We note that when $b = -1/\kappa$ the number $a + 1 \neq 0$ for the intervals of variation of α and κ under consideration, while the residual term must be replaced by $0\ln(\tau^2 A_0^2)$ with $\alpha = 1/(1+\nu) + \kappa/2(\kappa+1)$. The constant Φ_1 does not depend on τ and has the form: 1) $-b < 2$,

$$
\Phi_1 = [1/(h+1)]F(-b,\ h+1,\ h+2,\ -1) + [1/(a+1)]F(-b,\ a+1,\ a+2,\ -1);
$$

2) $2 \leqslant -b < 3$,

$$
\Phi_1 = -2^{b+2}(b+1)^{-1} + (h+1)^{-1}(b+1)^{-1}(b+h+2)F(-b-1,
$$
$$
h+1,\ h+2,\ -1) + (a+1)^{-1}(b+1)^{-1}(b+a+2)F(-b-1,\ a+1,\ a+2,\ -1);
$$

3) $-b \geqslant 3$; $m = [-b] - 1$ ([ ] is the integral part of the quantity),

$$\Phi_1 = -\frac{2^{b+2}}{b+1} - \sum_{k=2}^{m} \frac{(b+h+2)\ldots(b+h+k)+(b+a+2)\ldots(b+a+k)}{(b+1)\ldots(b+k)} \times$$
$$\times 2^{b+k} + \frac{(b+h+2)\ldots(b+h+m+1)}{(b+1)\ldots(b+m)} \frac{1}{h+1} F(-b-m, h+1, h+2, -1) +$$
$$+ \frac{(b+a+2)\ldots(b+a+m+1)}{(b+1)\ldots(b+m)} \frac{1}{a+1} F(-b-m, a+1, a+2, -1),$$

where $F(\alpha, \beta, \gamma, -1)$ are converging hypergeometric series [9]. Possible singular cases are included in 1-3) by the following agreement: if $a + j + 1 = 0$, $j = 0, 1, \ldots$, then the j-th term in the corresponding series is replaced by zero.

Using (11) and (13), for ζ from the region of joining we have

$$\left(\frac{x_2}{\tau}\right)^{1+\nu} - y_2^{1+\nu} = x_{20}^{1+\nu}(\xi, \tau) - y_{20}^{1+\nu}(\xi) + \Phi_2 + 0(\varepsilon),$$

$$\Phi_2 = 2^{\nu-1} c_0 \xi^{-\frac{1}{\varkappa}} \frac{1+\nu}{\alpha(1+\nu)-1} (F_0\tau)^{\frac{1+\nu}{\alpha(1+\nu)-1}} \tau^{-\frac{2}{\varkappa}} \left[\Phi_1\left(-\frac{1}{\varkappa}\right) + \sum_{j=1}^{l} \frac{\eta(\eta-1)\ldots(\eta-j+1)}{j!} \left(B\tau^{2-\frac{2}{\varkappa}}\right)^j \Phi_1\left(-\frac{j+1}{\varkappa}\right)\right]. \quad (14)$$

The expressions $[\alpha - 1/(1+\nu)]^{-k} \tau^{(1+\nu)/[\alpha(1+\nu)-1]}$, where the integer $k < l$, which are encountered here behave in a nonsingular way as $\alpha \to 1/(1+\nu)$ as the function $k! (\ln \tau)^{-k} \tau^{(1+\nu)/[\alpha(1+\nu)-1]}$. The residual term in (14) has the order $0(\tau^2 \ln \tau)$ when $\alpha = 1/(1+\nu) + \kappa/2(\kappa+1)$ and $0\left(\int_0^{\zeta} \frac{1}{\rho_0} d\zeta \tau^{\beta_0}\right) \sim \tau^{\frac{\varkappa(1+\nu)}{(\varkappa+1)(\alpha(1+\nu)-1)}}$ in the joining region in the remaining cases.

From a comparison of the orders of magnitude of the function Φ_2 and the residual term it follows that for $1/(1+\nu) < \alpha \leqslant 1/(1+\nu) + \kappa/2(\kappa+1)$ in the determination of the function x_{20} the function Φ_2 must be included in the residual term. As one can show for $1/(1+\nu) + \kappa/2(\kappa+1) < \alpha \leqslant (3+\nu)/2(1+\nu)$ the principal term ε_0 of the function $\tau^2 z_2$ of the first approximation in the joining region is of the same order of magnitude as the residual term in (14), while for $1/(1+\nu) \leqslant \alpha \leqslant 1/(1+\nu) + \kappa/2(\kappa+1)$ it does not exceed $0(\tau^2)$. Thus, it is incorrect to make the residual term in (14) concrete without allowance for the solution of the first approximation of the expansion (7). Therefore, setting $x_{20}(\xi, \tau) = y_{20}(\xi)$ in the case of $1/(1+\nu) \leqslant \alpha \leqslant 1/(1+\nu) + \kappa/2(\kappa+1)$ and setting $x_{20}^{1+\nu}(\xi, \tau) = y_{20}^{1+\nu}(\xi) - \Phi_2$ and $l = \left[\left(2 - \frac{2}{\varkappa}\right)^{-1}\left(\frac{2}{\varkappa} - \frac{1+\nu}{(\varkappa+1)(\alpha(1+\nu)-1)}\right)\right]$ in the case of $1/(1+\nu) + \kappa/2(\kappa+1) < \alpha/(3+\nu)/2(1+\nu)$ (in this case for $l = 0$ one must keep only the first term in the function Φ_2), we obtain in the joining region $(x_2/\tau)^{1+\nu} = y_2^{1+\nu} + 0(\varepsilon, \varepsilon_0)$. In the case of $\alpha = (3+\nu)/2(1+\nu)$ the expression for Φ_2 is simplified in accordance with (12) and has the form

$$\Phi_2 = -2^{\nu} c_0 \xi^{-\frac{1}{\varkappa}} \tau^{-\frac{2}{\varkappa}} \mu \left(\frac{\varkappa}{\varkappa-1} + \sum_{j=1}^{l} \frac{\eta(\eta-1)\ldots(\eta-j+1)}{j!} \left(B\tau^{2-\frac{2}{\varkappa}}\right)^j \frac{\varkappa}{\varkappa-j-1}\right). \quad (15)$$

The functions x_1 and y_1 are joined in accordance with the behavior of the function y_{20}. Taking $x_{10} = y_{10}(\xi)$ in (6), we obtain the following:

A. $y_{20} \neq 0$. In the joining region

$$x_1 = y_{10}(\xi) + y_{20}^{-\nu} \frac{d}{d\xi}\left(\frac{d}{d\xi}(y_{20}) \frac{dy_{10}^{-1}}{d\xi}\right)\zeta + 0(\varepsilon_1)\zeta, \qquad x_1 - y_1 = 0(\varepsilon_1)\zeta.$$

The quantity $0(\varepsilon_1)$ does not exceed in order of magnitude the maximum of τ^{β_0}, $\tau^{2-2/\kappa}$, $\tau^{\frac{l\varkappa(1+\nu)}{(\varkappa+1)(\alpha(1+\nu)-1)} - \frac{2}{\varkappa+1}}$. Thus, with an accuracy $0(\varepsilon_1)\tau^{\frac{\varkappa(1+\nu)}{(\varkappa+1)(\alpha(1+\nu)-1)}}$ in comparison with unity $x_1 = y_1$ in the joining region.

B. $y_{20} = 0$. 1) $\nu = 0$. In the joining region

$$x_1 = y_{10}(\xi) + 0\left(\tau^{\frac{1+\nu}{\alpha(1+\nu)-1}-\frac{2}{\varkappa}}\right)\zeta + 0\left(\int\limits_0^{\zeta}\int \frac{1}{\rho}\,d\zeta d\zeta\right),$$

and, as one can show,

$$x_1 - y_1 = 0\left(\tau^{\frac{2\varkappa(1+\nu)}{(\varkappa+1)(\alpha(1+\nu)-1)}-\frac{2}{\varkappa+1}},\quad \tau^{\frac{\varkappa(1+\nu)}{(\varkappa+1)(\alpha(1+\nu)-1)}+\frac{2}{\varkappa+1}}\right).$$

2) $\nu = 1$. In the joining region

$$x_1 = y_{10}(\xi) + \xi^{\frac{1}{2\varkappa}}\frac{d}{d\xi}\left[\frac{d}{d\xi}\left(\xi^{-\frac{1}{2\varkappa}}\right)\frac{dy_{10}^{-1}}{d\xi}\right]\zeta + 0\left(\tau^{2-\frac{2}{\varkappa}},\quad \tau^{\beta_0}\right)\zeta,$$

$$x_1 - y_1 = 0(\varepsilon_1)\tau^{\varkappa(1+\nu)/(\varkappa+1)(\alpha(1+\nu)-1)}.$$

After the arbitrary functions x_{20} and x_{10} are determined, one can verify that the joining of the remaining functions is performed with an accuracy $0[\tau^{2/(\kappa+1)}]$ in comparison with unity, while a solution which is uniformly valid in the region under consideration is constructed as follows. In the outer region the solution is described by the zeroth approximation of the expansion (7), while in the inner region it is described by the functions

$$\left(\frac{x_2}{\tau}\right)^{1+\nu} = x_{20}^{1+\nu}(\xi,\tau) + 2^{\nu}\int\limits_0^{\zeta}\frac{1}{\rho f}\,d\zeta,\quad u_1 = f,$$

$$x_1 = y_{10}(\xi) + x_{11}\zeta,\quad \frac{u_2}{\tau}\frac{\partial x_1}{d\xi} = f\frac{\partial}{\partial\xi}\left(\frac{x_2}{\tau}\right),\quad 1/\rho = c_0\xi^{-1/\varkappa}\tau^{-2/\varkappa}X$$

(the functions x_{20} and x_{11} were determined above).

The calculation of the integral is simplified by the fact that the pressure enters into the integrand as a parameter, and it is actually performed during the determination of the arbitrary function x_{20}. Here the outer and inner regions, as shown above, have a common part where both expansions are applicable. The shape of the surface $\zeta = 0$ of the body has the form

$$\left(\frac{x_2}{\tau}\right)^{1+\nu} = y_{20}^{1+\nu}(\xi) + \begin{cases} 0\left(\tau^{\frac{\varkappa(1+\nu)}{(\varkappa+1)(\alpha(1+\nu)-1)}},\ \varepsilon_0\right), & \frac{1}{1+\nu} < \alpha < \frac{1}{1+\nu} + \frac{\varkappa}{2(\varkappa+1)},\\ -\Phi_2, & \frac{1}{1+\nu} + \frac{\varkappa}{2(\varkappa+1)} < \alpha \leqslant \frac{3+\nu}{2(1+\nu)},\end{cases}$$

$$x_1 = y_{10}(\xi).$$

Thus, if one considers only the zeroth approximation of the expansion (7), then as $\alpha \to 1/(1+\nu)$ the correction to the contour of the body obtained from the solution of the corresponding unsteady problem becomes as small as desired in the region under consideration. The transverse size of the region of inapplicability of the zeroth approximation of the expansion (7), determined above by the values of $\zeta \leqslant 0[\tau^{\kappa(1+\nu)/(\kappa+1)(\alpha(1+\nu)-1)}]$, also becomes as small as desired, which reflects the fact of the continuous dependence of the solution on the boundary conditions, since when $\alpha = 1/(1+\nu)$ the entire field of flow is described by the zeroth approximation [1]. However, for all $1/(1+\nu) < \alpha \leqslant (3+\nu)/2(1+\nu)$ the correction to the zeroth approximation of the expansion (7) in the inner region is not small.

In comparing particular cases of the solution obtained with known solutions we note that if the shape of the front in the vicinity of the apex has the form $x_2/\tau = \kappa_1 x_1^{2/(3+\nu)}$, then the results of [10] are suitable for the determination of the shape of the surface of the body. And, as follows from [10], the equation of the surface of the body has the form

$$\left(\frac{x_2}{\tau}\right)^{1+\nu} = Y_{b0}^{1+\nu}(x_1) + p_{00}^{-\frac{1}{\varkappa}}(x_1)\frac{\varkappa}{\varkappa+1}\left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa}}\left(\frac{2}{3+\nu}\right)^2\varkappa_1^{3+\nu}\tau^{2\left(\frac{\varkappa-1}{\varkappa}\right)},$$

where Y_{b0} is the shape of the body and p_{00} is the pressure distribution over the body, which follow from the solution of the corresponding unsteady problem. In this case, in [10] terms of order $O(\tau^{4[(\kappa-1)/\kappa]})$ are discarded in the calculation. One can verify that this case corresponds to the following values of the parameters used above: $\kappa_1 = 2^{\nu/2} c^{-1/\alpha(1+\nu)}$, $\alpha = (3 + \nu)/2(1 + \nu)$. Retaining in (15) only the term of order $O(\tau^{2(\kappa-1)/\kappa})$ and using the expression obtained above for the shape of the body, we obtain

$$\left(\frac{x_2}{\tau}\right)^{1+\nu} = y_{20}^{1+\nu}(\xi) + \frac{\varkappa}{\varkappa+1}\left(\frac{2}{\varkappa+1}\right)^{\frac{1}{\varkappa}}\left(\frac{4}{9}\right)^{1-\nu} c^{-2}\xi^{-\frac{1}{\varkappa}}\tau^{2-\frac{2}{\varkappa}}, \quad x_1 = y_{10}(\xi).$$

Considering the meaning of the functions y_{20} and y_{10} and the connection $c = c(\kappa_1)$, we see that the two expressions fully coincide. Here the terms of higher order of smallness than the first term in (15) are due to the allowance for the departure of the function f from unity. As follows from (15), the allowance for the latter makes sense for values of κ near unity.

In the case when the shape of the front everywhere has the form $x_2 = c_1 x_1^n$ and $K = \infty$ (the self-similar case in the corresponding unsteady problem), the solution as $x_1 \to \infty$ is constructed in [3]. Let us make a comparison, taking $c_1 = 2^{\nu/2} c^{-n}\tau$, $n = 1/\alpha(1 + \nu)$, and being limited for simplicity to values of $1/(1 + \nu) + \kappa/2(\kappa + 1) < \alpha < (3 + \nu)/2(1 + \nu)$ and to the region of $\zeta \leqslant (\tau F_0)^{(1+\nu)/[\alpha(1+\nu)-1]}$ (for ζ from this region the projection of the vector of the unit normal to the front onto the x_1 axis is equivalent to unity in order of magnitude). The departure of u_1, the longitudinal component of the velocity of the disturbed stream, from unity was neglected in the determination of the coordinate x_2 in [3]. In the present report the departure was taken into account by the function f and entered into the function Φ_2 in the form of the terms additional to the first term. It is therefore appropriate to make the comparison by taking $f = 1$ and leaving only the first term in the expression for Φ_2.

From the results presented above we get the form for the function x_2 in the indicated region:

$$\frac{x_2}{\tau} = y_b(x_1) - \xi_b^{-\frac{1}{\varkappa}} y_b^{-\nu} c_0 \tau^{-\frac{2}{\varkappa}}\left(A_1 - \zeta F\left(\frac{1}{\varkappa}, \frac{k}{2}, \frac{k}{2}+1, -\tau^{-2}F_0^{-2}\zeta^{\frac{2}{k}}\right)\right),$$

$$A_1 = (F_0\tau)^k (F(1/\varkappa,\ k/2,\ k/2+1, -1) + [\varkappa k/(2-\varkappa k)]\, F(1/\varkappa, 1/\varkappa - k/2, 1/\varkappa - k/2 + 1, -1));$$

where $k = [\alpha - 1/(1+\nu)]^{-1}$; $y_b = 2^{\nu/2} c^{-1/\alpha(1+\nu)} \lambda_0 x_1^{1/\alpha(1+\nu)}$ is the surface of the body and $\xi_b = \frac{2^{1-\nu}}{\varkappa+1}\alpha^{-2} c^{-\frac{2}{\alpha(1+\nu)}} h_0 x_1^{-2\left(1-\frac{1}{\alpha(1+\nu)}\right)}$ is the pressure distribution over the body which follows from the solution of the unsteady problem; λ_0 and h_0 are constants. The corresponding expression from [3], rewritten in the notation of the present report, has the form

$$\frac{x_2}{\tau} = y_b(x_1) - \xi_b^{-\frac{1}{\varkappa}} y_b^{-\nu} c_0 \tau^{-\frac{2}{\varkappa}}\left((F_0\tau)^k N - \zeta F\left(\frac{1}{\varkappa}, \frac{k}{2}, \frac{k}{2}+1, -\tau^{-2}F_0^{-2}\zeta^{\frac{2}{k}}\right)\right).$$

Here N is the product of the gamma functions

$$N = \Gamma(k/2 + 1)\Gamma(1/\varkappa - k/2)\Gamma^{-1}(1/\varkappa).$$

Using the well-known properties of hypergeometric series, one can show that $A_1 = (F_0\tau^k)N$. Therefore, the expressions for x_2 fully coincide.

The pressure distribution which follows from the transformation of the function $x_1 = x_1(\xi, \zeta)$ has the form

$$p = \tau^2 \frac{2^{1-\nu}}{\varkappa+1}\alpha^{-2} c^{-\frac{2}{\alpha(1+\nu)}} x_1^{-\frac{2}{k\alpha}}\left(h_0 + \frac{\varkappa+1}{2k}\lambda_0^{2-\nu} c^{\frac{1}{\alpha}} x_1^{-\frac{1}{\alpha}}\zeta\right).$$

The latter coincides with the corresponding expression of [3] with the accuracy of terms containing the functions P_2 and P_4 (notation of [3]). The function $P_2 \sim O(\tau^2)$ is determined, as follows from [3], by the solution of the first approximation of the expansion (7). And the allowance for the term P_4 makes a contribution to the pressure distribution no larger than $O(\tau^{4-2/\kappa}\zeta)$. In the present report neither of these corrections was taken into account in the derivation of the expression for x_1. The form of the remaining gasdynamic quantities u_1 and u_2 is fully determined by the distributions of the functions x_2, p, and ρ, and therefore, no comparison is made for them.

In the general case the agreement established above allows one to effectively construct a solution to the inverse steady problem any time that a solution is constructed in the corresponding unsteady one-dimensional problem.

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SEMIEMPIRICAL THEORY OF THE GENERATION OF DISCRETE TONES BY A SUPERSONIC UNDEREXPANDED JET FLOWING OVER AN OBSTACLE

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§1. The phenomenon of the generation of a strong discrete tone by a supersonic underexpanded jet flowing over an obstacle was first discovered by Hartmann [1]. There are presently a considerable number of reports devoted to the experimental study of the Hartmann effect [2-4]. However, the mechanism of formation of these oscillations has not been clarified up to now [2, 5]. An elementary theory of this phenomenon is presented in the present report.

A diagram of a supersonic underexpanded jet flowing over a flat obstacle is presented in Fig. 1 (1 is the jet boundary; 2 is the central compression shock (the Mach disk); 3 is the suspended shock; 4 is the reflected shock; and 5 is the contact discontinuity). The effect consists in the fact that the flow becomes unstable at certain values of the nozzle Mach number M_a, degree of nonratedness $n = p_a/p_s$ of the jet (p_a is the pressure in the jet

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